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Paradoxes of Denotation

I. Introduction

There are finitely many phrases of English that have fewer than nineteen syllables. Among these are phrases that define integers - for example, "the positive square root of thirty six". So there are finitely many integers defined by English phrases with fewer than nineteen syllables. Since there are infinitely many integers, there must be an integer which is *the least integer not definable in fewer than twenty syllables*. The italicized phrase has nineteen syllables - and this phrase defines an integer. So the least integer not definable in fewer than twenty syllables is definable in nineteen syllables.

This is Berry's paradox. It is one of several so-called paradoxes of definability - the paradoxes due to Richard and König are other prominent members of this family.[1] In referring to these paradoxes as paradoxes of definability, I am following tradition. But the traditional terminology may mislead in two ways. First, the appearance of a modal element in the term \`definable' is deceptive: we might just as well have presented Berry's paradox in terms of the phrase \`the least integer not defined in less than nineteen syllables'. Second, the paradoxes do not turn on any technical sense of definition. When we generate these paradoxes, we count as a definition any phrase that denotes or refers to a number; so, for example, the phrase \`the number of planets' will count as a definition of the number 9.[2] The paradoxes turn on the semantic relation that holds between a referring expression and its referent, whether the relation is expressed by \`defines' or \`denotes' or \`refers to'. The paradoxes would be better called paradoxes of reference, or paradoxes of denotation. I shall refer to them as paradoxes of denotation - but throughout this paper, the term \`denotes' can be taken as interchangeable with \`refers to', or \`defines' understood in the appropriately broad sense.

Since their discovery early this century, the paradoxes have played an important role in the development of modern set theory. They show that the move from a predicate to its extension is not always straightforward. Take, for example, the predicate \`integer definable in fewer than twenty syllables' - if we suppose that this predicate determines a set, we are led to a contradiction. Zermelo, and later Skolem and von Neumann among others, provided axiomatizations of set theory that prohibit such predicates.[3]

Despite their impact on set theory, paradoxes of definability are traditionally classified as semantical rather than logical or set-theoretical.[4] A logical paradox turns on mathematical or set-theoretical notions; Russell's paradox, for example, turns on the notions of set and membership. In contrast, a semantic paradox makes ineliminable use of semantic terms from ordinary, non-mathematical language -- the term `denotes' in the cases of Berry, Richard and König, and the term `true' in the case of the Liar.

This contrast should be kept in mind when we try to solve a semantic paradox. Semantic paradoxes are generated by our everyday semantic notions. So we are after an account of these ordinary notions - an account of familiar English predicates, like `refers to' or `denotes' or `true'. The problem is not primarily a formal or technical one: we are concerned first and foremost with natural languages like English, not with regimented formal languages. One way we can take the point to heart is by attending to our uses of terms like `denotes' and `true' - in particular, to the reasoning we conduct with these terms. Now it is well-known that there are `strengthened' versions of the Liar, and I believe that a careful examination of the associated strengthened reasoning throws light on our concept of truth.[5] In contrast, strengthened versions of the paradoxes of denotation have received little or no attention[6] - but the associated strengthened reasoning also promises to illumine our concept of reference.

In this paper, I propose a new resolution of the paradoxes of denotation - what I shall call a *singularity solution*. In line with the considerations of the previous paragraph, my proposal pays close attention to natural reasoning involving the predicate `denotes'. I start out with a simple paradox of denotation. In the course of treating this paradox I present the main ideas of the singularity proposal, and draw a sharp contrast between this proposal and the orthodox Tarskian approach. I go on to apply the singularity approach to Berry's paradox. In the final section I consider the bearing of the singularity proposal on the issue of semantic universality. The solution I propose to these paradoxes of denotation is of a piece with the solution I have offered elsewhere to the Liar.[7] My hope, then, is to articulate here an approach that applies to semantic paradoxes generally.

II. A simple paradox of denotation

Suppose that at noon 3/27/93 I write on the board the following expressions:

A. the ratio between the circumference and diameter of a circle.
B. the positive square root of 36.
C. the sum of the numbers denoted by expressions on the board in room 101 at noon 3/27/93.

I believe that room 101 is the room next door, and that written on the board there are expressions that refer to numbers. But I am mistaken about my whereabouts -- I am in fact in room 101. It is clear that A and B denote numbers. But what number does C denote? We can reason as follows:

> Suppose towards a contradiction that C denotes a number, say k. Then the sum of the numbers denoted by A, B and C is π+6+k. But this number is the number denoted by C; so k=k+π+6, which is a contradiction. So C is a pathological referring expression: it appears to denote a number, but it does not, on pain of contradiction.

From this reasoning, we infer:

(1) C is pathological, and does not denote a number.

We can now strengthen our reasoning, building on our conclusion that C is pathological. We can argue as follows:

> Suppose that C is indeed pathological, and does not denote a number. Then A and B are the only expressions on the board that denote numbers. So the sum of the numbers denoted by expressions on the board in room 101 at noon 3/27/93 is π+6. But in the previous sentence there occurs a token of the same type as C, call it $C^{/}$. Now we infer:

(2) $C^{/}$ - a token of the same type as C - denotes π+6.

In a way characteristic of strengthened reasoning, we have found that we can use the words of a pathological referring expression to refer to a number.[8] This calls for explanation. We might suppose that we have on our hands a strengthened paradox, and try to block the reasoning in some way -- after all, C and $C^{/}$ are composed of the very same words with the same linguistic meaning, and yet one denotes a number and the other doesn't. However, the reasoning is intuitive, and appears to be valid. We should not block it by artificial, ad hoc means. Since the reasoning is natural, I would rather regard it as data that expresses semantic intuitions we have about the notion of denotation or reference. My strategy will be to find a plausible analysis that preserves the validity of the reasoning, and respects the data.

III. `Denotes' as a context-sensitive term

Since there appear to be no *semantic* differences between the tokens C and C', it is natural to look for *pragmatic* differences. There are a number of differences between the context of C and the context of C'. Clearly there are differences of time and place; C' is uttered later, and, unlike C, C' is not written on the board in room 101. And if we suppose that someone other than myself carries out the strengthened reasoning, there is a difference of speaker too. Still, the familiar contextual parameters of speaker, time, and place do not tell the whole story.

A fourth difference is that C and C' are embedded in different stages of the strengthened discourse. We may split the discourse into two stages, the first culminating in (1), and the second leading from (1) to (2). The token C belongs to the first stage, and the token C' to the second. In general, the correct interpretation of an expression or a stretch of discourse may depend on the larger discourse in which it is embedded.[9] The reasoning from (1) to (2) is second not merely in temporal order, but in logical order as well. The second stage starts out from a subconclusion, namely (1), established by the first stage of the argument. We can think of the second stage as *reflective* with respect to the first: at the second stage of the reasoning, we reflect on the pathological nature of C, established by the first stage. This logical order constitutes a difference in the relation that each stage of the discourse bears to the discourse as a whole.

A fifth difference is found in speaker's intentions. When I first utter C, I do not intend to produce a pathological utterance, but rather to pick out the expressions written on the board next door. This intention is overridden, given the time and place of my utterance: I have unwittingly landed myself in paradox. Throughout the second stage of the reasoning, the reflective stage, we have a very different intention, to treat C as a pathological referring expression and see where this leads us. This intention is not overridden. On the contrary, the fact that we intentionally take C to be pathological leads to the conclusion that C' denotes a number.

There is a sixth difference between the two stages, a shift of relevant information. When I first utter C, the information that C is pathological is not available to me. But this information is available throughout the reflective second stage of the reasoning: it is precisely what is established by the reasoning at the first stage. The reasoning of the second stage should be interpreted as incorporating this information.[10]

So we distinguish two contexts, one in which I write C, and the second in which we produce C'. Call these *the original context* and *the reflective context* respectively. Between these contexts there is a shift in a number of contextual parameters: speaker, time, place, discourse position, intention, and relevant information. Now we want to explain the fact that C does not denote a number (conclusion 1) while C' does (conclusion 2), yet C and C' appear to be semantically indistinguishable. A pragmatic explanation is indicated, one that takes account of the shift in the contextual parameters. If we accept the appropriateness of a pragmatic explanation, then we should expect to find a term occurring in C and C', and in (1) and (2), that is context-sensitive. When we inspect the terms occurring in these expressions, there is only one plausible candidate: the predicate `denotes'. I propose that, in the absence of any reasonable alternative, we take the predicate `denotes' to be the context-sensitive term.

Let `$\text{denotes}_{OR}$' abbreviate `denotes in the original context', and let `$\text{denotes}_{RE}$' abbreviate `denotes in the reflective context'. Let us turn first to the representation of C. We take the occurrence of `denotes' in C to be sensitive to the context in which it occurs.[11] So C is analyzed as:

(C) the sum of the numbers denoted_{OR} by expressions on the board in room 101 at noon 3/27/93.

At the first stage, we go on to reason that C is pathological. In the course of this reasoning, there is an implicit use of what we can call a <u>denotation schema</u>. A denotation schema has this form:

s denotes n iff p is identical to n,

where instances of the schema are obtained by substituting for `p' any referring expression, for `s' any name of this expression, and for `n' any name of a number. We suppose that C denotes k. The instantiation of the schema for C and k is:

C denotes k iff the sum of the numbers denoted by expressions on the board in room

101 at noon 3/27/93 is k.

In our reasoning, we assume the left hand side of the biconditional, infer the right hand side, and obtain a contradiction - since the sum of the numbers denoted by A, B and C is π+6+k. But this contradiction is forthcoming only if the denotation schema is the schema for the occurrence of `denotes' in C. That is, the schema we have utilized in our reasoning is the `denotes_{OR}' schema: we derive a contradiction via the biconditional

C denotes_{OR} k iff the sum of the numbers denoted_{OR} by expressions on the board in room 101 at noon 3/27/93 is k.

A denotation schema provides denotation conditions for a given referring expression - that is, the conditions under which the phrase denotes a number.[12] At the first stage of the reasoning, we give the denotation conditions for denoting expressions - in particular C - via the `denotes_{OR}' schema. We find that C does not have denotation_{OR} conditions, and conclude that C does not denote_{OR} a number.

Now, at the second stage, we reason that A and B are the only expressions on the board that denote_{OR} numbers, since C does not. A and B denote π and 6 respectively. So we infer that the sum of the numbers denoted_{OR} by expressions on the board in room 101 at noon 3/27/93 is π+6. In producing $C^{/}$ here, we have in effect repeated C. But we have repeated C in a new reflective context, in which we no longer provide denotation conditions via the `$\text{denotes}_{OR}$' schema. Instead, we provide denotation conditions for $C^{/}$ via the `denotes_{RE}' schema - and $C^{/}$ <u>does</u> have denotation_{RE} conditions. Both sides of the biconditional

$C^{/}$ denotes_{RE} k iff the sum of the numbers denoted_{OR} by expressions on the board in room 101 at noon 3/27/93 is k

are true for k = π+6. $C^{/}$ denotes_{RE} π+6.

C and $C^{/}$ are indeed <u>semantically</u> indistinguishable. The difference between them is purely pragmatic. It is a matter of the denotation schema by which C and $C^{/}$ are given denotation conditions.[13] At the first stage of the reasoning, C is assessed via the `$\text{denotes}_{OR}$' schema; at the second stage, $C^{/}$ is assessed via the `denotes_{RE}' schema. The schema that provides denotation conditions is determined by the contextual parameters. And the shift in discourse position, intentions, and information produces a change of implicated schema. At the second stage, the schema that provides denotation conditions is reflective with respect to the denoting phrase C. That the schema is reflective in this way is a product of the reasoning of the first stage, the assessment of C as a pathological defining phrase, and the intention to treat C as such.

According to our analysis, then, (1) is represented by

C does not denote$_{OR}$ a number,

and (2) by

$C^{/}$ denotes$_{RE}$ π+6.

(1) and (2) are both true, and are the result of valid reasoning. Notice that if we assess C via the `denotes$_{RE}$' schema, we find that C, like $C^{/}$, denotes$_{RE}$ $\pi$+6; and if we assess $C^{/}$ via the `denotes$_{OR}$' schema, we find that $C^{/}$, like C, does not denote$_{OR}$ a number. Both C and $C^{/}$ have denotation$_{RE}$ conditions; neither have denotation$_{OR}$ conditions. So the predicates `denotes$_{OR}$' and `denotes$_{RE}$' have different extensions. For C and $C^{/}$ are not in the extension of `denotes$_{OR}$' (more precisely, neither C nor $C^{/}$ are the first member of any ordered pair in the extension of `denotes$_{OR}$'). But C and $C^{/}$ are in the extension of `denotes$_{RE}$' (more precisely, the ordered pairs $<C,\pi+6>$ and $<C^{/},\pi+6>$ are in the extension of `denotes$_{RE}$'). So `denotes' is a context-sensitive term that shifts its extension according to context.[14]

IV. A singularity proposal

Thus far, I have argued that in the course of our strengthened reasoning the denotation predicate undergoes a change in extension with the shift in context. On our analysis, neither C nor $C^{/}$ are in the extension of `denotes$_{OR}$', but both are in the extension of `denotes$_{RE}$'. So C and $C^{/}$ are excluded from the extension of `denotes$_{OR}$'. What else is excluded from the extension of `denotes$_{OR}$'? And what is the relation between the extensions of `denotes$_{OR}$' and `denotes$_{RE}$'?

A possible response here is a Tarskian one: when we move from the first stage of the reasoning to the second, we push up a level of language. The predicate `denotes$_{OR}$' is the denotation predicate of the language of the first stage; the predicate `denotes$_{RE}$' is the more comprehensive denotation predicate of the semantically richer language of the second stage. On such a hierarchical account, the extension of `denotes$_{OR}$' is a proper subset of the extension of `denotes$_{RE}$'.

Some kind of Tarskian resolution is perhaps the orthodoxy regarding definability paradoxes.[15] But the Tarskian approach offers too rationalized an account of our natural language, in particular the predicate `denotes'. Surely English does not contain infinitely many distinct denotation predicates, but just one.[16] And surely the stratification of English into a hierarchy of distinct languages is highly artificial. With Tarski, we may doubt

> whether the language of everyday life, after being `rationalized' in this way, would still preserve its naturalness and whether it would not rather take on the characteristic features of the formalized languages.[17]

Further, it is hard to see how levels can be assigned to occurrences of the denotation predicate in any systematic way. How are we to interpret a given phrase containing the predicate `denotes'? To which language does it belong? Except in very simple cases, we will have little basis for an assignment of one level rather than another. And what level should we assign to a global statement like `Every expression of English either denotes a number or does not denote a number'? Any assignment of a level here will compromise the global nature of the statement.

The account I shall offer is in a strong sense *anti-hierarchical*. The leading idea is that in pathological defining phrases, there are *minimal* restrictions on occurrences of `denotes'. At this point, a pragmatic principle of interpretation comes into play: the principle of *Minimality*. According to Minimality, restrictions on occurrences of `denotes' are kept to a minimum: we are to restrict the application of `denotes' only when there is reason to do so.

Suppose you say: "The number denoted by `the square of 1' is 1". Here, your use of `denotes' is quite unproblematic. Should C be excluded from its extension? Minimality says no. And this is surely plausible. As we've seen, <u>within</u> its context of utterance, C is pathological; but outside that context, C denotes a number. Since your utterance is quite unrelated to C, we have no reason to interpret your utterance as in some way pathologically linked to C. Assessed from outside its context, C does denote a number (namely, π+6), and so we have no reason to withhold C from the extension of `denotes' in your utterance. It would be a poor interpretation that implicated your utterance in semantic pathology. In general, speakers do not usually aim to produce pathological utterances, or utterances implicated in paradox.[18] By adopting Minimality, we respect this pragmatic fact.

Further, if we adopt Minimality we respect a basic intuition about predicates. Intuitively, we take a predicate to pick out everything with the property that the predicate denotes. In general, if an individual has the property denoted by the predicate φ, then that individual is in the extension of φ. The more restrictions we place on occurrences of `denotes', the more we are at odds with this intuition. We do expect any solution to a genuine paradox to require some revision of our intuitions. But the more a solution conflicts with our intuitions, the less plausible that solution will be.

For example, the Tarskian stratification of the denotation predicate involves massive restrictions on occurrences of `denotes'. On a standard Tarskian line, the referring expression `the only even prime' is of level 0; your unproblematic denoting phrase ("the number denoted by `the square of 1'") is of level 1, and so on, through the levels. Your use of `denotes' in an utterance of level 1 has in its extension all referring expressions of level 0, *and no others*. So all sentences of level 1 and beyond are excluded from the extension of such a use of `denotes'. Gödel remarks of Russell's type theory that "...each concept is significant only ... for an infinitely small portion of all objects."[19] A similar complaint can be made about a standard Tarskian account of denotation: an ordinary use of `denotes' will apply to only a fraction of all the denoting expressions.

Minimality keeps surprise to a minimum: our uses of `denotes' apply to almost all denoting phrases. We are sometimes forced to restrict `denotes' - we must, for example, limit the extension of `denotes' in C by excluding C itself. Still, according to Minimality, we exclude only those denoting expressions that cannot be included.

So my proposal identifies what I shall call *singularities* of the concept *denotes*. In general, if a defining phrase σ cannot be given denotation$_{c\alpha}$ conditions, then σ is a *singularity* of `denotes$_{c\alpha}$'. And if, further, the schema for `denotes$_{c\alpha}$' is the one by which σ is assessed in its context, then σ is *pathological*. So C is a singularity of `denotes$_{OR}$', and C is pathological too, since in its context C is assessed by the schema for `denotes$_{OR}$'. $C^{/}$ is also a singularity of `denotes$_{OR}$', but $C^{/}$ is *not* pathological, since in its context $C^{/}$ is assessed by the schema for

`denotes_{RE}', and $C^{/}$ does denote_{RE} a number.

C is a singularity only in a context-relative way - there is an appropriate reflective context in which C is in the extension of `denotes'. In the case of our strengthened discourse, we *can*, from our subsequent reflective context, assess C as a denoting phrase that denotes a definite number, namely $\pi$+6. It is only within its context of utterance that C is a singularity of `denotes'.[20]

When we fix extensions of occurrences of `denotes' in accordance with Minimality, we fix those extensions in a way that is not determined by what the speaker, or we as interpreters, *know*. Such ignorance does not affect the general way in which extensions are determined by Minimality. The relevant empirical and semantical facts may not be available to the speaker or interpreter; nevertheless, these facts in part determine the extension of `denotes'.

No occurrence of `denotes' is without singularities. For example, suppose you utter an innocent enough denoting phrase: "the number denoted by `the square of 1'". But you perversely continue: "plus the number denoted by `the square of 2', plus the sum of the numbers denoted by phrases in this utterance." Within your utterance, there is no shift to a reflective context. So if your first use of `denotes' is represented by `$\text{denotes}_U$', so are the subsequent uses. Now the final definite description token in your utterance (the token that begins `the sum of...') is a pathological denoting phrase - like C, it includes itself within its own scope. We cannot give this pathological denoting phrase denotation_U conditions, on pain of contradiction. And so this token is a singularity of `$\text{defines}_U$'. It may be that there are no *actual* phrases uttered that force restrictions on a given occurrence of `denotes'; there may be no *actual* singularities. But continuations like yours here are always *possible* - and they yield singularities of the given occurrence of `denotes'.

We are now in a position to see the anti-Tarskian nature of the singularity proposal. As a consequence of Minimality, the singularity proposal is not hierarchical. Consider the strengthened discourse that led to (2). We concluded that $C^{/}$ denotes π+6. Suppose we now add:

(a) And so the number denoted by $C^{/}$ is irrational.

This addition contains a defining phrase token, call it $C^{//}$. Given the context, $C^{//}$ is represented as: "the number denoted_{RE} by $C^{/}$". By Minimality, $C^{//}$ is not excluded from the extension of `denotes' in C (or in $C^{/}$): $C^{//}$ is not a singularity of `denotes' in C. $C^{//}$ is an unproblematic referring expression in its context of utterance, and it remains so when assessed from any other context. In particular, the attempt to give $C^{//}$ denotation_{OR} conditions succeeds. Given the biconditional

$C^{//}$ denotes_{OR} π+6 iff the number denoted_{RE} by $C^{/}$ is π+6

and the truth of the right hand side, it follows that $C^{//}$ denotes_{OR} π+6. So in the extension of `$\text{denotes}_{OR}$' there is a referring expression in which the predicate `denotes_{RE}' appears. For the Tarskian, this would amount to an unacceptable mixing of language levels. On the singularity proposal, there are no such levels.

Moreover, there are singularities of `$\text{denotes}_{RE}$' that are not singularities of `denotes_{OR}'. As we noted two paragraphs back, every occurrence of `denotes' has singularities, and the occurrence of `denotes' in (2) is no exception. Consider an addition to (2) more perverse than (a):

(b) And so the number denoted by $C^{/}$, plus the number denoted by `the square of 2', plus the sum of the numbers denoted by phrases in this utterance, is irrational.

Given the context, the occurrences of `denotes' in our continuation will be represented by `denotes_{RE}'. The final definite description token in our utterance (beginning `the sum of') is pathological - and since we cannot give it $\text{denotation}_{RE}$ conditions, it is a singularity of `denotes_{RE}'.

But now, just as reflected on C, so we can reflect in turn on this pathological token. And in a suitable reflective context we can conclude that it denotes (π+6)+4 - since the only denoting phrases in (b) that denote_{RE} numbers are the first two phrases, and the sum of the numbers denoted_{RE} by these phrases is (π+6)+4. Within its context of utterance, the token does not denote a number - but outside that context it denotes a number, by Minimality. In particular, the token denotes_{OR} (π+6)+4. By Minimality, then, this token is not a singularity of `$\text{denotes}_{OR}$'. On a Tarskian account, the extension of `denotes_{OR}' will be a proper subset of the extension of `denotes_{RE}'. On the singularity proposal, neither extension includes the other.

It seems to me an unwarranted Tarskian prejudice to insist that in moving from the first stage of the strengthened reasoning to the second we move to an essentially richer language. Perhaps this prejudice is encouraged by consideration of the utterer's epistemic situation. In the context of utterance of C, I do not know or believe that C is pathological. I may come to know or believe this, but only by transcending the epistemic situation I am in when I produce C. But as we have already said, the extension of a defining expression is not determined by what speakers know.

Further, the Tarskian cannot help herself to this epistemic justification of the levels. For it would be quite possible for someone to produce intentionally a pathological token on the board, in full knowledge that she will go on to reflect on that utterance, qua pathological. We may be quite self-conscious about the production of pathological referring expressions; this is so, for example, when we discuss paradoxes of denotation. We may produce pathological referring expressions in full knowledge that when assessed in another (reflective) context, these tokens will denote a number, because pathological in the present context. In this case, there is no shift in what is known, yet still the Tarskian will discern a shift in language levels.[21]

Minimality allows room for any degree of semantic awareness. We might even be omniscient, able to survey the extension of any occurrence of `denotes'. According to Minimality, a use of `denotes' applies to all denoting expressions except those that are pathological in the given context, whatever the epistemic state of the speaker. In particular, the tokens in (a) and in (b) each denote$_{OR}$ a number, whether I know it or not in my original context of utterance.

So on my proposal, our simple paradox is to be treated by the identification and exclusion of singularities.[22] We treat everyday English not as a hierarchy of languages, but as a single language. We do not divide up the concept of denotation between infinitely many languages; rather we identify singularities of a single, context-sensitive denotation predicate.

Gödel noted that Russell's theory brings in a new idea for the solution of the paradoxes:

> It consists in blaming the paradoxes ... on the assumption that every concept gives a meaningful proposition, if asserted for any arbitrary object or objects as arguments.[23]

Gödel goes on to say that the simple theory of types carries through this idea on the basis of a further restrictive principle, by which objects are grouped into mutually exclusive ranges of significance, or types, arranged in a hierarchy.

Gödel suggests that we reject this principle, while retaining the idea that not every concept gives a meaningful proposition for any object as argument:

> It is not impossible that the idea of limited ranges of significance could be carried out without the above restrictive principle. It might even turn out that it is possible to assume every concept to be significant everywhere except for certain `singular points' or `limiting points', so that the paradoxes would appear as something analogous to dividing by zero. Such a system would be most satisfying in the following respect: our logical intuitions would then remain correct up to certain minor corrections, i.e. they could then be considered to give an essentially correct, only somewhat `blurred', picture of the real state of affairs.[24]

I take my singularity proposal to be very much in the spirit of Gödel's remarks. And we can claim for it the same satisfying feature: our logical intuitions about `denotes' are almost correct. It is only in pathological or paradoxical contexts that we may mistakenly suppose that certain phrases denote when they do not -- and in such cases our applications of `denotes' require only minimal corrections. A second intuition that requires revision is that `true' is a predicate constant. Strengthened discourses indicate that `denotes' shifts its extension according to context. But these shifts are kept to a minimum.

In correcting both these intuitions, we retain a single denotation predicate which undergoes minimal changes in its extension according to context. There is no wholesale revision of the notion of denotation; no division of `denotes' into infinitely many distinct predicates; no splitting of everyday English into an infinite hierarchy of languages.

V. Denotation trees and the identification of singularities

So far we have spoken informally of pathological denoting expressions and singularities, largely by way of examples. In this section, I shall sketch a way in which we might make the account rather more general.

Consider again our simple paradox. Suppose we represented the token C as an ordered pair <type(C), denotes_{OR}>, where the first element is the type of C, and the second indicates the representation of the occurrence of `denotes$_{OR}$' in C. Then something is missing. This representation does not distinguish C from $C^{/}$, yet the former denoting expression is pathological and the latter is not. There is something more to consider: the schema by which the token is given denotation conditions. In its context of utterance, C is assessed via the schema for `denotes$_{OR}$', and this is what renders it pathological. So we can represent C more perspicuously as an ordered triple - <type(C), denotes$_{OR}$, denotes$_{OR}$> - where the third element indicates the schema by which C is assessed in its context of utterance. Let us call this the *primary representation* of C. In general, the *primary representation* of a denoting expression token involving `denotes' indicates in order the type of the token, the occurrence of `denotes' in the token, and the implicated schema by which the token is assessed in its context. The primary representation of $C^{/}$ is the triple <type(C), denotes$_{OR}$, denotes$_{RE}$>, since in its context of utterance, $C^{/}$ is assessed by the schema for `denotes$_{RE}$'.

We can also assess C from contexts other than its own. For example, we can assess C from the subsequent reflective context. We may represent such an assessment as the triple <type(C), denotes$_{OR}$, denotes$_{RE}$>. We can think of this as a *secondary representation* of C, since here C is not assessed by its implicated schema. Notice that this secondary representation of C is identical to the primary representation of $C^{/}$. This is appropriate, since both C and $C^{/}$ denote$_{RE}$ the same number.

We have said that C is a pathological defining token. Let us now represent its pathological character in a rather more rigorous way, via the notion of *ungroundedness*. Some denoting expressions do not make references to other denoting expressions; the denoting expression `the square root of four' is like this. Some denoting expressions do make reference to other denoting expressions, for example, the denoting expression `the number denoted by `the square root of four''. But this denoting expression is unproblematic, because it makes reference to a denoting expression that does not make reference to a denoting expression. We may iterate, and obtain increasingly deeply nested denoting expressions, starting with the denoting expression, `the number denoted by "the number denoted by `the square root of four''". All denoting expressions in this sequence are unproblematic because they may be ultimately traced back to a denoting expression that does not make reference to denoting expressions. Such denoting expressions are, intuitively, *grounded*. But other denoting expressions are *ungrounded*. For example, C makes reference to itself, and so in tracing back through the denoting expressions to which C makes reference, we never escape denoting expressions that make reference to denoting expressions.

In a moment, we will represent the ungroundedness of a denoting expression like C via a certain kind of tree. But we need to prepare the ground a little. Let us first introduce the notion of a *determinant* of a denoting expression. Consider the token C. How do we determine a referent for this token in its context of utterance? We must first determine the referents of certain defining phrases to which C refers - A, B, and C. These are the defining phrases that determine a referent for C in its context of utterance. We will call these phrases the <u>determinants</u> of C. Consider now $C^/$. To determine a referent for $C^/$ in its context of utterance, we consider A and B only, since in the reflective context C is explicitly excluded as a determinant. A and B are the determinants of $C^/$. In general, the identification of the determinants of a denoting expression is guided by both semantic and pragmatic considerations. On the semantic side, we consider the denoting phrases to which the expression refers. We take these to be the determinants unless there are overriding contextual considerations, as in the example of $C^/$: in the reflective context of $C^/$, one of the phrases to which $C^/$ refers is explicitly rejected as a determinant, because it is pathological. We have already noted that there is no semantic difference between C and $C^/$ - but there is a pragmatic difference. And with this contextual shift, there is a shift in the determinants.

It's important to notice that C purports to denote a number in terms of what its determinants <u>*denote*$_{OR}$</u>, since the occurrence of `denotes' in C is represented by `denotes$_{OR}$'. So to determine the number (if any) that C denotes, we need to determine what numbers are denoted$_{OR}$ by C's determinants. To determine the number denoted by C, then, the appropriate schema by which to assess these determinants is the `denotes$_{OR}$' schema. More generally, let the primary representation of a defining phrase σ be the triple <type(σ), denotes$_{c\alpha}$, denotes$_{c\beta}$>. Let ρ be a determinant of σ. To determine the number (if any) that σ denotes, we should assess ρ via the schema for `denotes$_{c\alpha}$'.

We will represent the ungroundedness of C via what we shall call its *primary denotation* *tree*. To construct the primary denotation tree for C, we start with the primary representation of C, the triple <type(C), denotes$_{OR}$, denotes$_{OR}$>. This is the node at the top of the tree. At the second tier are the determinants of C, suitably represented. A and B contain no context-sensitive terms, and so these are suitably represented via their types - we need not worry about the schema by which they are assessed.[25] This isn't so for the other determinant of C, namely C itself. Following the remarks of the previous paragraph, C is to be assessed via the schema for `denotes$_{OR}$'. Accordingly, we represent C at the second tier as <type(C), denotes$_{OR}$, denotes$_{OR}$>. This is the primary representation of C again, which in turn generates a third tier of nodes. And so on, indefinitely. The primary tree for C looks like this:

<type(C), denotes$_{OR}$, denotes$_{OR}$>
/ | \\
type(A) type(B) <type(C), denotes$_{OR}$, denotes$_{OR}$>
/ | .
type(A) type(B) .
.

This tree has an infinite branch, *on which the primary representation of C repeats*. This indicates that C is an ungrounded defining phrase. The repetition of the primary representation shows that C cannot be given denotation conditions by the schema for `denotes$_{OR}$' - and so we can also say that C is a *singularity* of `denotes$_{OR}$'.

In general, we construct the primary denotation tree for a defining phrase σ as follows. The top node is the primary representation of σ - let it be <type(σ), denotes$_{c\alpha}$, denotes$_{c\beta}$>. At the second tier are the determinants of σ, suitably represented in line with the remarks of the last two paragraphs. Branches lead from each of these determinants to their own determinants. And so on.

We can now give a more general characterization of the notions of *ungroundedness* and *singularities*. Let σ be a denoting phrase token. Suppose σ's primary representation repeats on an infinite branch of σ's primary denotation tree. Then σ is *ungrounded*. Further, if the primary representation is <type(σ), denotes$_{c\alpha}$, denotes$_{c\beta}$>, then σ is a *singularity* of `denotes$_{c\beta}$' - the repetition of the primary representation indicates that σ cannot be given denotation conditions by the schema for `$\text{denotes}_{c\beta}$'.[26]

Consider now C'. The determinants of C' are just A and B, and the primary denotation tree for C' is:

```
          <type(C), denotes_OR, denotes_RE>
                /              \
          type(A)              type(B)
```

This well-founded tree indicates the groundedness of C'. The number denoted by C' depends on A and B only.[27]

Suppose now we assess C from some neutral context, quite unrelated to the contexts of C and C'. Let the secondary representation of C here be the triple <type(C), denotes_{OR}, denotes_{N}>. What are the determinants of C under this representation? Since the neutral context provides no overriding pragmatic considerations, the determinants are the denoting expressions to which C makes reference, namely A, B, and C. So the secondary denotation tree for C is this:

```
      <type(C), denotes_OR, denotes_N>
      /           |               \
type(A)        type(B)     <type(C), denotes_OR, denotes_OR>
                            /           |             \
                      type(A)        type(B)     <type(C), denotes_OR, denotes_OR>
                                                   /          |             .
                                             type(A)       type(B)           .
                                                                              .
```

Now this tree has an infinite branch. But the secondary representation does <u>not</u> repeat on this branch: the assessment of C from outside C's context stands above the circle in which the primary representation of C is caught.

We can give further expression to this idea via the notion of a *pruned* tree. We *prune* the tree here by terminating the infinite branch at the first occurrence of a non-repeating node.[28] The pruned tree is:

<type(C), denotes$_{OR}$, denotes$_N$>
/ \
type(A) type(B)

The pruned tree indicates that C can be assessed by the schema for `denotes$_N$': C denotes$_N$ a number in terms of A and B. So C is *not* a singularity of `denotes$_N$', *in accordance with Minimality*.

VI. Berry's paradox

Let us return to Berry's paradox, expressed now in terms of the notion of denotation. We can present two stages of reasoning quite parallel to the reasoning about C.

> Let p be *the least integer not denoted in fewer than nineteen syllables*. Let D be the italicized phrase that occurs in the previous sentence. If D does indeed denote p, then p is denoted by a phrase with fewer than nineteen syllables, and we obtain a contradiction. We conclude:

$(1')$ D is a pathological denoting phrase, and does not denote a number.

At the second stage, we argue:

> Since D is a pathological denoting phrase, it is not among the phrases of English with fewer than nineteen syllables that denote numbers. Once the problematic phrase D is eliminated (along with other pathological phrases), we are left with the English phrases that do denote numbers. And so there is a number - let it be m - which is *the least integer not denoted in fewer than nineteen syllables*. The italicized phrase in the previous sentence - call it D' - is of the same type as D. We infer:

$(2')$ D' - a token of the same type as D - does denote a number m.[29]

The analysis of this reasoning follows the same pattern as before. Again, we distinguish two contexts, the original context in which D is produced, and the subsequent reflective context in which D' is produced. We treat `denotes' as a context-sensitive term. Let `denotes$_O$' and `denotes$_R$' abbreviate `denotes in the original context' and `denotes in the reflective context' respectively. Then D and $D'$ are both analyzed as `the least integer not denoted$_O$ in fewer than nineteen syllables'. $(1')$ is represented by

D does not denote$_O$ a number,

and ($2'$) is represented by

D' denotes$_R$ a number.

In its context of utterance, D is assessed via the \`denotes$_O$' schema; and D does not have denotation$_O$ conditions. So D is a singularity of \`denotes$_O$' (as is D'). In its context of utterance, D' is assessed via the \`denotes$_R$' schema. D' does have denotation$_R$ conditions, and does denote$_R$ a number (as does D).

As with our simple paradox, with the shift in context there is a shift in the determinants. Notice that when we set up the Berry, when we speak of the phrases of English that have fewer than nineteen syllables and that denote numbers, we distinguish phrases of English as phrase-*types* - this guarantees that the number of denoting phrases that have fewer than nineteen syllables is finite. These phrase-types are all determinants of D. Now there is one further determinant that is explicitly included - D itself. And the strengthened reasoning indicates that we cannot simply consider D *qua* phrase-type. D is a token that does not denote a number, while D' is a token of the same type that does denote a number. So when we identify D as one of its own determinants, we do so *qua* token (or phrase-type in a context) and not as a phrase-type *simpliciter*. When we turn to the determinants of D', we see that in the reflective context of D', D is explicitly excluded, since it is pathological. The determinants of D' are just the unproblematic phrase-types, and D' does define a number. Just as C is a determinant of C but not of C', so D is a determinant of D but not of D'.

We can put all this in terms of denotation trees. The primary representation of D is the triple <type(D), denotes$_O$, denotes$_O$>. We can represent the primary denotation tree for D as follows:

```
            <type(D), denotes_O, denotes_O>
            /              |              \
phrase-type      phrase-type     ...  <type(D), denotes_O, denotes_O>
                                       /             |              \
                             phrase-type   phrase-type ...   <type(D), denotes_O, denotes_O>
                                                              /   |     ...              .
                                                                                           .
                                                                                             .
```

Since the primary representation of D is a repeating node, D is ungrounded. Further, D is a singularity of `denotes$_{O}$'.

The primary representation of $D^{/}$ is <type(D), denotes$_{O}$, denotes$_{R}$>. Again, the determinants of $D^{/}$ are the phrase-types of English with fewer than nineteen syllables that denote numbers: D is explicitly excluded. The primary denotation tree for $D^{/}$ is:

<type(D), denotes$_{O}$, denotes$_{R}$>
/ | …
phrase-type phrase-type …

The number m denoted by $D^{/}$ is determined by these unproblematic phrase-types.

Thus far, the analysis of the Berry looks very like that of our simple paradox. But there is a way in which the Berry has more staying power. While $C^{/}$ is not itself an expression on the board, $D^{/}$ *is* itself a phrase of English. This means that we can extend our strengthened reasoning in a way that we could not in the case of the simple paradox.

> Given ($2^{/}$), $D^{/}$ is a phrase of English with fewer than nineteen syllables that denotes a number. So *the least integer not denoted in fewer than nineteen syllables* is a number different from m. Let $D^{//}$ be the italicized phrase in the previous sentence. Then

($3^{/}$) $D^{//}$ - a token of the same type as $D^{/}$ - denotes a number different from m.

Again, rather than blocking the reasoning by *ad hoc* measures, we should seek an analysis that treats (3) as a true conclusion reached by valid reasoning. And again, a contextual analysis is called for. Since it follows on from ($2^{/}$), the first part of the reasoning is analyzed as follows:

> Given ($2^{/}$), $D^{/}$ is a phrase of English with fewer than nineteen syllables that denotes$_{R}$ a number. So *the least integer not denoted$_{R}$ in fewer than nineteen syllables* is a number different from m.

So the occurrence of `denotes' in $D^{//}$ is represented by `denotes$_{R}$'. It follows that $D^{//}$ does not denote$_{R}$ a number. Just as $D^{/}$ cannot be assessed by the denotation$_{0}$ schema, so $D^{//}$ cannot be assessed by the denotation$_{R}$ schema - if we insert $D^{//}$ into the schema for `denotes$_{R}$', we obtain a contradiction. $D^{//}$ is a singularity of `denotes$_{R}$'. If we are to represent ($3^{/}$) as a true conclusion, we cannot represent the occurrence of `denotes' in ($3^{/}$) as `denotes$_{R}$'. The context of ($3^{/}$) must be *reflective* with respect to $D^{//}$. Just as the context of $(2^{/})$ is reflective with respect to singularities of `denotes$_0$', in particular D and $D^{/}$, so the context of $(3^{/})$ is reflective with respect to singularities of `denotes$_R$', in particular $D^{//}$. Let `denotes$_{RR}$' abbreviate `denotes in the further reflective context of $3^{/}$'. Then $D^{//}$ is assessed via the schema for `denotes$_{RR}$', and (3) is represented as `$D^{//}$ denotes$_{RR}$ a number different from m'.

The primary representation of $D^{//}$ is <type(D), denotes$_R$, denotes$_{RR}$>. The context makes it clear that the determinants of $D^{//}$ are the phrase types of English that denote numbers, together with $D^{/}$, which is explicitly included. In the primary denotation tree for $D^{//}$, we will represent $D^{/}$ as assessed by the schema for `denotes$_R$', since `denotes$_R$' occurs in $D^{//}$. So the triple <type(D), denotes$_O$, denotes$_R$> appears as a node at the second tier. This is the primary representation of $D^{/}$, and so the primary denotation tree for $D^{/}$ will appear as a subtree. The primary denotation tree for $D^{//}$ is given by:

<type(D), denotes$_R$, denotes$_{RR}$>
/ | …
phrase-type phrase-type … <type(D), denotes$_O$, denotes$_R$>
/ | …
phrase-type phrase-type …

The number denoted by $D^{//}$ - a number different from m - is ultimately determined by the unproblematic phrase-types; these terminate every branch.

$D^{//}$ is an unproblematic denoting token. When assessed by its implicated schema, $D^{//}$ does denote a number. And, by Minimality, $D^{//}$ retains its unproblematic status when assessed from contexts other than its own.[30] For example, by Minimality $D^{//}$ denotes$_O$ a number: $D^{//}$ is not a singularity of `denotes$_O$'. As we've remarked, there is an exception: $D^{//}$ <u>is</u> a singularity of `denotes$_R$'.[31]

We can keep going in the same way. We can reason about $D^{//}$ just as we reasoned about $D^{/}$, and produce a token of the same type in a new reflective context. This new token will be a singularity of `denotes$_{RR}$', but not of `denotes$_R$' or `denotes$_O$', by Minimality. It is, then, a consequence of Minimality that each of the predicates `denotes_O', `$\text{denotes}_R$' and `denotes_{RR}' has singularities that the others do not. So when we produce further reflective tokens of the type "the least integer not denoted in fewer than nineteen syllables", we do not ascend a Tarskian hierarchy. Take any two members of the corresponding sequence of denotation predicates: neither is more or less comprehensive than the other. This remains so however far we go along the sequence.

We resolve the Berry as we resolved our simple paradox of definability. According to our resolution, the predicate `denotes' is a context-sensitive predicate that is minimally restricted on any given occasion of use. The solutions to König's and Richard's paradoxes are exactly parallel - for details, see the Appendix.[32]

VII. Semantic Universality

Natural languages are remarkably flexible and open-ended. If there is something that can be said, it might seem that a natural language like English has at least the potential to say it. Natural languages evolve; they always admit of extension, of increased expressive power. Tarski speaks of the "all-comprehensive, universal character" of natural language, and continues:

> The common language is universal and is intended to be so. It is supposed to provide adequate facilities for expressing everything that can be expressed at all, in any language whatsoever; it is continually expanding to satisfy this requirement.[33]

In particular, Tarski says, natural languages are *semantically universal*. According to Tarski's characterization, a semantically universal language contains names for its own expressions, contains its own semantical predicates that apply to expressions of the language, and has the resources for describing the proper use of these expressions.[34] In short, a semantically universal language can say everything there is to say about its own semantics. Now a natural language like English *does* appear to be semantically universal.[35] Consider, for example, the predicate `denotes'. This is an English predicate applying to expressions of English (strictly speaking, to ordered pairs of English expressions and their referents) - and prima facie, to exactly those expressions of English that are denoting expressions. The application of `denotes' appears to be *global*, in the sense that it applies to every denoting expression of English. And we seem to be able to describe the proper use of this term in English. It is the same for the other semantic terms of English. So it seems to me that we should respect as far as possible Tarski's intuition that natural languages like English are semantically universal.

I think that the singularity proposal goes a long way to accommodate this intuition. An occurrence of the context-sensitive predicate `denotes' is as close to universal as it can be without contradiction - it applies to all denoting expressions except its singularities. Moreover, according to the singularity proposal, even denoting expressions that are singularities relative to a given context fall into the extensions of `denotes' in other contexts (such as an appropriate reflective context). So the application of any occurrence of `denotes' is almost global, and those denoting expressions that prevent its application from being fully global are captured by other uses of `denotes'.

We can take these points a little further. Many have thought that the goal of a universal language is unattainable, because any theory of the semantical predicates of a language must be couched in an essentially richer metalanguage.[36] But if we adopt the singularity proposal, then we are not driven to this conclusion. Let L be that fragment of English that is free of context-sensitive terms. Let $L^{/}$ be the result of adding to L the context-sensitive predicate `denotes'. We can think of $L^{/}$ as the object language for our singularity account. Now in this paper I have not attempted to give a formal theory of `denotes'. I have only described some notions - e.g. denotation tree, groundedness, and singularity - that I take to be central to such a theory. But suppose for a moment that we have a formal singularity theory.[37]

With any theory of context-sensitive terms, there is an inevitable separation of the object language and the language of the theory. We can take the language of the theory to be a classical formal language which quantifies over contexts, and in which context-sensitive terms do not appear. Unlike the object language, the language of the theory is context-independent. Now the language of the theory - call it T - will be in certain ways richer than the object language. For example, T will contain the predicate `expression of $L^{/}$ that denotes in some context'. This predicate is the denotation predicate for $L^{/}$, a predicate that applies to exactly the denoting expressions of $L^{/}$. It will include in its extension all those expressions of $L^{/}$ that denote a number in some context - and among these expressions are the singularities. Given that any occurrence of the context-sensitive predicate `denotes' has singularities, the predicate `expression of L' that denotes in some context' will be in this way more inclusive than any occurrence of the predicate `denotes' of L. This may tempt us to suppose that the denotation predicate for $L'$ is more comprehensive than any occurrence of `denotes' in the object language, and that T is a Tarskian metalanguage for L'. But the temptation should be resisted. Let us see why.

In the language T, we can formulate denoting expressions that cannot be formulated in L' - for example, any denoting expression that uses the predicate for L'.[38] But, by Minimality, these denoting expressions will *not* be excluded from the extension of any occurrence of `denotes' in $L'$. Only the singularities of the given occurrence are excluded - and unproblematic denoting expressions expressed in T will not be identified as singularities. (Intuitively, the theory tells us what is *excluded from* the extension of occurrences of `denotes', *not* what is included; we take a `downward' route rather than an `upward' route.) This shows that T is not a Tarskian metalanguage for L', since ordinary context-sensitive uses of `denotes' apply to denoting expressions that can be formulated in T but not in L'. Again, according to the singularity proposal, paradox is avoided not by a Tarskian ascent, but by the identification and exclusion of singularities.

Moreover, if we suppose that T is a classical formal language, free of context-sensitive terms, then T cannot contain its own denotation predicate, on pain of contradiction.[39] So we can generate from this formal language a Tarskian hierarchy of formal languages, each containing the denotation predicate for the preceding language. But none of the denoting expressions expressible in these languages are identified as singularities, and so none are excluded from the extensions of our ordinary context-sensitive uses of `denotes'. To speak metaphorically, our uses of `denotes' arch over not only the denoting expressions of T, but also all the denoting expressions expressed by the languages of this hierarchy.

So we do not take the formal hierarchy generated from T to explicate our concept of denotation. The levels do not correspond to any stratification of the context-sensitive predicate `denotes'. The singularity proposal abandons this Tarskian route. For the Tarskian, questions about the extent of the hierarchy and quantification over the levels will present special difficulties. Of course, these are substantial questions, quite independently of any particular proposal about `denotes' in English. But they present no special difficulty for the singularity account.

According to the singularity account, an ordinary context-sensitive use of `denotes' applies `almost everywhere', failing to apply only to those denoting expressions that are pathological in its context of utterance. By Minimality, when we use `denotes' we point to as many denoting phrases as we can point to from our context of utterance. These denoting expressions include those expressible in T, and at any level of the Tarskian hierarchy which can be generated from that theory (whatever the extent of the hierarchy).

To return to semantic universality. With respect to the concept of denotation, a semantically universal language would have a term applying to *all* the denoting phrases of the language. Such a language would be subject to the paradoxes of denotation. According to the singularity account, any use of `denotes' has its singularities. But a use of `denotes' applies everywhere else - even to denoting expressions couched in the language T of the theory, and the hierarchy of languages generated from T. Indeed, a use of `denotes' would apply to any denoting expression of any language, as long as the denoting expression is not identified as a singularity. If we adopt the singularity proposal, then any use of `denotes' is as close to a global denotation predicate as it can be. In this way we respect Tarski's intuition that we can say everything semantical there is to say.

Appendix: König's and Richard's paradoxes

We analyze these paradoxes along lines exactly parallel to the analysis of Berry's paradox.

König's paradox runs as follows. There are only denumerably many English definitions of ordinal numbers, but non-denumerably many ordinals. Consider the ordinals that are not denoted by any English expression. There must be a least among these ordinals. But then the expression `the least ordinal not denoted by any English expression' denotes this number, and we have a contradiction.

As with the Berry, there is strengthened reasoning associated with König's paradox. For economy, I present the two stages of the reasoning with the contextual subscripts in place, where these subscripts indicate, as before, the original context in which the pathological denoting phrase is produced, and the reflective context in which we reflect on the pathological phrase. For the intuitive reasoning ignore the subscripts.

> Let r be *the least ordinal not denoted$_{or}$ by an English phrase*. Let K be the italicized phrase that occurs in the previous sentence. Then r is denoted$_{or}$ by an English phrase, and this is a contradiction. So we conclude:

(1^K) K is a pathological denoting phrase, and does not denote$_{or}$ an ordinal. At the second stage, we argue:

> Since K is a pathological denoting phrase, it is not among the phrases of English that denote$_{or}$ ordinals. Once the problematic phrase K is eliminated, then we are left with the English phrases that do denote$_{or}$ ordinals. And so there is an ordinal which is *the least ordinal not denoted$_{or}$ by an English phrase*. The italicized phrase in the previous sentence, call it $K^{/}$, is of the same type as K. We infer:

(2^K) $K^{/}$ - a token of the same type as K - does denote$_{re}$ an ordinal.

The analysis here carries over from the Berry point for point. In particular, K is a singularity of the occurrence of `denotes' in K, as the primary denotation tree for K indicates.

In exactly parallel fashion, we can present the strengthened reasoning associated with Richard's paradox in two stages. Again, for the intuitive reasoning ignore the subscripts.

> Obtain an enumeration of all the English phrases that denote$_o$ a real number. Let w be *the real which has 0 for its integral part, and 1 in its nth decimal place if the number denoted$_o$ by the nth phrase in the enumeration does not have 1 in its nth decimal place, and which has 2 in its nth decimal place if the number denoted$_o$ by the nth phrase has 1 in its nth decimal place*. Let R be the italicized phrase that occurs in the previous sentence. Since R is a diagonal definition, w is different from itself, and we have a contradiction. We infer:

(1^R) R is a pathological denoting phrase, and does not denote$_o$ a real.

At the second stage we argue:

> Since R is a pathological denoting phrase, it is not among the phrases of English that denote$_o$ real numbers. Once the problematic phrase R is eliminated, we are left with the English phrases that do denote$_o$ real numbers. And so there is a number which is *the real*

which has 0 for its integral part, and 1 in its nth decimal place if the number denoted$_o$ by the nth phrase in the enumeration does not have 1 in its nth decimal place, and which has 2 in its nth decimal place if the number denoted$_o$ by the nth phrase has 1 in its nth decimal place. The italicized phrase in the previous sentence, call it $R^/$, is of the same type as R. We infer:

(2^R) $R^/$ - a token of the same type as R - does denote$_r$ a real.

In the same way as before, R is a singularity of the occurrence of `denotes' in R, as the primary definition tree for R indicates. Again, the paradox is resolved by the identification and exclusion of singularities.

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Endnotes

1. Berry's paradox is reported in Russell 1906 and 1908. For König's paradox, see König 1905; and for Richard's paradox, see Richard 1905.

2. Richard and König themselves used the notion of definability in this broad way, as coextensive with the notions of denotation or reference. For more on this, see the present author's 1993b.

3. Zermelo required that the property denoted by the predicate be definite, that it either apply or not apply to any individual from the given domain (see Axiom III in Zermelo 1908). Predicates like `integer definable in less than nineteen syllables' failed to meet this condition. Skolem and von Neumann found Zermelo's condition too vague, and offered more precise restrictions (see Skolem 1922 and von Neumann 1925).

4. Peano urged this distinction in connection with Richard's paradox. According to Peano, the paradox

> contains ideas of the `common language', ideas very familiar to us but not determinate, and they cause all the ambiguity. Richard's example does not belong to mathematics but to linguistics; an element that is fundamental in the definition of N cannot be defined in an exact way (according to the rules of mathematics). From an element that is not well-defined, we can draw several mutually contradictory conclusions. (Peano 1906, in Peano 1957 pp. 357-358. My thanks to Loredana Ziff for help with the translation of Peano's paper.)

Ramsey drew the distinction between the two types of paradox in Ramsey 1925.

5. Discussions of the so-called Strengthened Liar can be found in Parsons 1974, Burge 1979, Barwise and Etchemendy 1987, Gaifman 1988 and 1992, and the present author's 1993a.

6. There are however hints of such strengthened reasoning in the work of Richard, Peano and Poincaré. For more on this, see the present author's 1993b and note 29 below.

7. See the present author's 1993a.

8. It is the same with the strengthened Liar reasoning: we can use the words of a pathological Liar sentence to make a true assertion. Suppose that at noon 3/28/93 I write on the board the following sentence:

(L) The sentence written on the board in room 101 at noon 3/28/93 is not true.

I believe that room 101 is the room next door, and that there is an untrue sentence written there. But in fact I am in room 101 - so the empirical circumstances conspire to make (L) a semantically pathological sentence. For if we assume that (L) is true, we may infer that (L) is not true; and if we assume (L) is false, then (L) is not true, and therefore (L) is true. So we conclude:

(1^*)   L is pathological.

But now we can strengthen our reasoning, building on our conclusion that (L) is pathological. Since (L) is pathological, (L) is not true. That is we can assert:

(2^*)   L is not true.

But now we have used the very words of (L), with same linguistic meaning, to make a true assertion. While (L) is pathological, (2^*) is true. Similarly, while (C) is pathological, ($C^/$) defines a number.

9. This has been emphasized by Paul Ziff, among others; see Ziff 1972, p. 36.

10. A shift of relevant information may not always occur in strengthened reasoning: I may intentionally produce a pathological denoting expression, in order to reflect upon it (see note 21 and the associated text).

11. Here and elsewhere I speak of an occurrence of `denotes' in an utterance, even if that occurrence is in the passive form.

12. Compare the more familiar truth schema:

s is true iff p,

where to obtain an instance we substitute for `p' a sentence, and for `s' a name of that sentence. The used sentence on the right hand side provides truth conditions for the mentioned sentence on the left.

13. Compare Burge's discussion of the corresponding Strengthened Liar reasoning in Burge 1979, section II.

14. We can give a similar analysis of strengthened Liar reasoning. Consider again the reasoning in note 8. We distinguish the original context in which (L) is uttered, and the reflective context in which we produce (2^*). The predicates `$\text{true}_{\text{orig}}$' and `$\text{true}_{\text{refl}}$' abbreviate `true in the original context' and `true in the reflective context' respectively. (L) and (2^*) are both analyzed as

(L) is not $\text{true}_{\text{orig}}$,

but while (L) is assessed by the schema for `$\text{true}_{\text{orig}}$', ($2^*$) is assessed by the schema for `$\text{true}_{\text{refl}}$'. In a way quite parallel to the case of C and $C^/$, neither (L) nor (2^*) are in the extension of `$\text{true}_{\text{orig}}$', but both are in the extension of `$\text{true}_{\text{refl}}$'. On this analysis, `true', like `denotes', is a context-sensitive term that shifts its extension according to context.

15. Jean van Heijenoort writes: "Today [Richard's] paradox is generally considered solved by the distinction of language levels" (in Jean van Heijenoort 1967, p. 142.)

16. Compare the analogous complaint about a Tarskian treatment of `true'; see, for example, Kripke 1975.

17. Tarski 1983, p. 267.

18. Of course, philosophical discussions of definability paradoxes provide exceptions to this general rule.

19. Kurt Gödel 1944, in Schilpp 1944, p. 149.

20. Some phrases will never get into the extension of `denotes', even after we have reflected on them. Suppose I write: `The number denoted by the phrase on the board is irrational'. If the phrase in question (`the number denoted by the phrase on the board') is the very one I have written, then it is pathological. But, unlike the case of C, the recognition that this phrase is pathological does not enable us to determine a referent for it. Such a pathological phrase is not a singularity of the concept of denotation, since, despite initial appearances, it does not denote at all - it is not a denoting phrase in any context.

21. In this case, there will not be a shift in relevant information between the context in which we produce the pathological token and the context in which we reflect upon it (contrast the case of C). But there will still be a difference in discourse position. And there will still be a difference in intentions: it is one thing to intend to produce a pathological token, and another to evaluate it qua pathological.

22. Elsewhere, I have suggested a similar treatment of the Liar paradox. See the present author's 1993a.

23. Kurt Gödel 1944, in Schilpp 1944, p. 228.

24. Op. cit., p. 229.

25. For simplicity, we will in general ignore all context-sensitive expressions other than `denotes'. So we will always represent a denoting phrase via its type unless it contains an occurrence of `denotes'.

26. For another example, consider a case of mutually referring phrases. Suppose you write on the board in room 102 just before noon on 3/29/92 these denoting phrases:
(a) the only even prime.
(b) the least perfect number.
(c) the product of the numbers denoted by expressions on the board in room 103 at noon 3/29/92.
And suppose that at the same time I write these denoting phrases on the board in room 103:
(d) the least odd prime.
(e) the cube root of 27.
(f) the product of the numbers denoted by expressions on the board in room 102 at noon 3/29/92.

In the usual way, we can argue to the conclusion that (c) and (f) are pathological denoting phrases. The cyclic character of (c) and (f) is captured by their respective denotation trees. Each tree has an infinite branch on which the primary representations of both (c) and (f) repeat. This indicates that (c) and (f) are each singularities of the occurrence of `denotes' in (c), and of the occurrence of `denotes' in (f).

Cases of 3-cycles, 4-cycles, ... n-cycles ... follow a similar pattern. We can also represent the ungroundedness of infinite chains of denoting phrases via denotation trees that contain infinite branches without repetitions.

27. Consider again the cycling case presented in the previous footnote. We could strengthen the reasoning that led to the conclusion that (c) and (f) are pathological, in the familiar way. For example, we could argue that since (f) is pathological, the only expressions on the board in room 103 that denote numbers are (d) and (e). And we may conclude that the product of the numbers denoted by expressions on the board in room 103 at noon 10/8/92 is 3x3. Our conclusion contains a token of the same type as (c), call it ($c^{/}$). Unlike (c), ($c^{/}$) is an unproblematic definition of the number 9. The primary definition tree for ($c^{/}$) contains two branches, leading from the primary representation of $c^{/}$ to each of (d) and (e): the number defined by $c^{/}$ is determined by the definitions (d) and (e) only.

28. In general, suppose we have a denotation tree all of whose infinite branches contain repeating nodes. Then the <u>pruned tree</u> is obtained by terminating each infinite branch at the first non-repeating node.

29. The final paragraph of Richard 1905 shows some sensitivity to this strengthened reasoning; see also Peano 1906 (esp. p. 357), Poincaré 1906, and the present author 1993b. According to Russell, the number defined by the phrase `the least integer not definable in less than nineteen syllables' is 111,777 (see Russell 1908, p. 153. There is something odd about this claim. Russell presumably arrives at this number because he is presupposing this standard list of English names of finite integers: `one', `two', `three', ... `one hundred and one'... . But then the paradox does not arise, since the problematic defining phrase is not a member of this list.

30. Suppose $D^{//}$ is assessed from another context, and let <type(D), denotes_R, denotes_M> be the appropriate secondary representation of $D^{//}$. The determinants of $D^{//}$ will remain the same, since they are unproblematic denoting expressions that are not subject to rejection as determinants. And the appropriate schema by which to assess these determinants is still the schema for `$\text{denotes}_R$', since it is the predicate `denotes_R' that appears in $D^{//}$. So, apart from the top node, $D^{//}$'s secondary denotation tree for $D^{//}$ will be just like its primary denotation tree. So $D^{//}$ denotes_M the same number that it denotes_R, and $D^{//}$ is <u>not</u> identified as a singularity of `denotes_M'. This accords with Minimality.

31. Assume that $D^{//}$ denotes_R a number. Now $D^{//}$ denotes a number in terms of denoting expressions that denote_R numbers, since `denotes_R' occurs in $D^{//}$. So one of the denoting expressions relevant to the determination of $D^{//}$'s referent will be $D^{//}$ itself - by our assumption that $D^{//}$ denotes_R a number, we count $D^{//}$ as one of its own determinants. It is easy to check that $D^{//}$ is a singularity of `denotes_R'.

32. For further discussion of Richard's paradox, see the present author's 1990 and 1993b.

33. Tarski 1969, p. 89. In the same vein, Tarski writes:

> A characteristic feature of colloquial language (in contrast to various scientific languages) is its universality. It would not be in harmony with the spirit of this language if in some other language a word occurred which could not be translated into it; it could be claimed that `if we can speak meaningfully about anything at all, we can also speak about it in colloquial language' (Tarski 1983, p. 164).

We should avoid misunderstandings about Tarski's notion of universality. To claim that natural languages are universal in Tarski's sense is not to claim that all concepts are expressible in natural language. This latter claim would be highly controversial, or perhaps just plain false. Consider, for example, the sets in the ZF hierarchy. For each set there is a distinct concept - say, being a member of that particular set. Given certain assumptions about natural languages (in particular, about upper limits on the size of vocabularies, and on the length of sentences), these concepts would outrun the expressive capacity of any natural language. But Tarski does not make the claim that natural languages can express all concepts. Rather, Tarski is claiming that if a concept is expressible in some language, then it is expressible in any natural language. This claim is perfectly compatible with the existence of concepts that are inexpressible (in every language).

34. Tarski uses the term `semantically universal' in Tarski 1969, p. 89, and `semantically closed' in Tarski 1944, p. 20.

35. This does not mean that we must accept that English is universal. We might take issue with Tarski's claim (quoted in note 33 above) that any word of a language may be translated into any natural language. Good translations, even adequate ones, are often hard to come by (for a discussion of this, see Dorit Bar-On 1993). Is this paper translatable into a semantically undeveloped natural language? Would such a language survive the morphological expansion required for such a translation, or would it have been superseded by another language? (Paul Ziff, for one, would reply to both these questions with an emphatic no: see Ziff 1988, p. 8).

36. See, for example, Kripke 1975, in Martin 1984, p. 79 and footnote 34.

37. In the present author's 1993a, such a theory is given for `true'.

38. For example, `the least positive integer denoted in some context by an expression of $L^{/}$'. A more interesting case may be provided by `the least ordinal number not denoted in any context by an expression of $L^{/}$'. If this phrase does denote an ordinal, it denotes an ordinal different from any denoting phrase of $L^{/}$. The denotation of this phrase will depend on cardinality considerations - in particular, the cardinality of the set of contexts, and the size of the vocabulary of $L^{/}$.

39. Suppose there is a predicate of T - `denotes in T' - that has in its extension all expressions of T that are denoting expressions. Then the expression `the least number not denoted in T' is itself an expression of T. And the contradiction associated with the Berry paradox is readily obtained.